Lab 3: Fourier-series

Grading

This Lab consists of four exercises. Once you have submitted your code in Matlab Grader AND once the deadline has past, your code will be checked for correctness. Note here, that upon submission, your code is already subjected to some basic checks that are aimed to verify whether your code will compile; these basics checks don't say anything about the correctness of your submission. You can visit Matlab Grader again after the deadline (give the servers some time to do all the assessments; this might even take a few days) to see how well you did. In case Matlab Grader indicates you failed an exercise, this does not automatically imply that you failed the entire exercise. Each exercise is subjected to n tests, where the number of tests can vary between exercises. In case Matlab Grader indicates you failed the exercise, this means that not all tests were passed (e.g. in an exercise with 7 tests, you could have passed 6 and Matlab Grader will indicate you failed the exercise). Your grade is calculated based on the number of tests you passed and not on the number of exercises you passed.

1 Introduction

1.1 Fourier-series

Periodical signals, which consist of a signal segment which is repeated continuously, are very common in electrical engineering. The time span of the signal segment is called the fundamental period T_0, and the fundamental frequency $f_0 = \frac{1}{T_0}$ describes the repetition speed of the signal segment. One might think of a clock signal in computers, the mains voltage, or test signals for electronic systems. Many 'signals' in daily life, such as sounds, also have a periodic nature, for example wind and string instruments or voiced sounds in speech.

All periodic signals can be decomposed in a constant plus an infinite number of sine and cosine functions, with frequencies which are integer multiples of the fundamental frequency $f_0 = \frac{1}{T_0}$. This sum is called a **Fourier-series**, which is a valuable aid in the analysis of signals and systems in the frequency domain. The response of a linear system, for example, can be described as the sum of all responses to separate sinusoidal input signals, using the superposition principle. Fourier-series can be analysed in multiple ways, as sinusoids, in an amplitude-phase form, and in a complex exponential form. The complex exponential form will be used in this Lab to show how separate terms of a Fourier-series can be analysed.

2 Overview

2.1 Definition of periodic functions

A signal $x(t)$, defined for all real values of t, is periodic if a positive value T_0 exists, so

$$x(t) = x(t + T_0), \tag{1}$$

where T_0 is the fundamental period of $x(t)$. It follows that in this case

$$x(t) = x(t + nT_0), \tag{2}$$

for all integer values of n. In case two signals $x(t)$ and $y(t)$ have the same fundamental period T_0, a linear combination $\alpha x(t) + \beta y(t)$ also has a fundamental period T_0.

2.2 Even and odd periodic functions

A signal $x(t)$ is called even if $x(t) = x(-t)$ for all t, and odd if $x(t) = -x(-t)$ for all t. Therefore, when $x(t)$ is even

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = 2\int_{0}^{\frac{T}{2}} x(t)\,dt, \tag{3}$$

and, when $x(t)$ is odd:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = 0. \tag{4}$$

The product of an even and an odd function results in an odd function.

2.3 Fourier-series

Using the following property, as explained in the lectures, a cosine signal can be reconstructed with two phasors:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{5}$$

Given a more commonly used notation, the equation can be written as:

$$\begin{aligned} A_k \cos(2\pi f_0 kt + \phi_k) &= \frac{A_k e^{j(2\pi f_0 kt + \phi_k)} + A_k e^{-j(2\pi f_0 kt + \phi_k)}}{2} \\ &= \frac{1}{2} A_k e^{j\phi_k} e^{j2\pi f_0 kt} + \frac{1}{2} A_k e^{-j\phi_k} e^{-j2\pi f_0 kt)} \end{aligned} \tag{6}$$

As can be seen, a cosine signal can be written as two phasors (time dependent complex exponents) with a positive and negative frequency, equal to the frequency for the cosine ($f_0 k$), multiplied with a complex amplitude (not time dependent), determined by the phase and the amplitude of the cosine signal. These complex amplitudes will from now on be denoted as α_k and thus we can write a real cosine signal as follows::

$$A_k \cos(2\pi f_0 kt + \phi_k) = \alpha_k e^{j2\pi f_0 kt} + \alpha_{-k} e^{-j2\pi f_0 kt} \tag{7}$$

with

$$\alpha_k = \frac{1}{2} A_k e^{j\phi_k} \text{ and } \alpha_{-k} = \alpha_k^* = \frac{1}{2} A_k e^{-j\phi_k} \tag{8}$$

During the lectures, we have seen that any real, periodic function $x(t)$ with fundamental period $f_0 = \frac{1}{T_0}$ either can be written as a the following series of sinusoidal signals:

$$x\left(t\right) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi}{T_0} kt + \phi_k\right), \tag{9}$$

or as the following series of complex exponentials (phasors):

$$x\left(t\right) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi}{T_0}kt} \tag{10}$$

with $\alpha_0 = A_0$ and α_k defined as in equation (8) for $k \neq 0$.

Equation (10) describes the Fourier series synthesis of a periodic function. On the other hand, if we want to analyse a periodic function we can use the following equation:

$$\alpha_k = \frac{1}{T_0} \int_0^{T_0} x\left(t\right) e^{-j\frac{2\pi}{T_0}kt} \mathrm{d}t. \tag{11}$$

Note: the boundaries of the integral do not necessarily range from 0 to T_0. The boundaries have to span one complete periode T_0 (e.g. $-T_0/4$ until $3T_0/4$ is also correct).

Exercise 1 [2 tests]

Given

$$\alpha_k = \begin{cases} \frac{4}{(\pi k)^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

create the signal

$$y(t) = \sum_{k=-9}^{9} \alpha_k e^{j2\pi f_0 kt}$$

with $f_0 = \frac{1}{T_0} = 1$ Hz. Plot the signal $y(t)$ on the time span [0, 2] seconds, with 201 measuring points in total. You may need to use equation (7).

2.4 Frequency spectrum

The frequency spectrum of a periodical signal $x(t)$ can be obtained by plotting the amplitude and phase, combined in one complex number, as a function of the frequency f[Hz]. The spectrum of a periodic signal hence consists of a series of discrete harmonics, given by a number of evenly spaced vertical lines, with a frequency spacing $f_0 = 1/T_0$. The vertical lines denote the complex number α_k which includes both magnitude A_k and phase ϕ_k respectively, for each of the separate phasors of equation (10).

Exercise 2 [6 tests]

Given the (incomplete) frequency spectrum of a real, periodic signal in Figure 1 below. Fill in the open spaces and create a signal $y_1(t)$ using Matlab on the time span $t_1 \in [0, 2.5]$. Next, convert the cosine in Figure 2 into a function $y_2(t)$ with the same time span as in the figure and add this signal to the existing signal $y_1(t)$, creating a new signal $y_3(t)$ (on a new time span, see below).

Make a script that creates three subplots underneath each other. The first plot should contain $y_1(t)$ on the specified time span [0, 2.5]. The second plot should contain $y_2(t)$ on a time span (t_2) which is the same time span as in Figure 2. The last plot should contain $y_3(t)$ on the time span $t_3 \in [0, ?]$ in such a way that exactly 2 full periods of the function $y_3(t)$ are visible. Use the solution template provided. Both The time spans t_1 and t_2 should contain 251 samples, while the time span t_3 should contain 1001 samples.

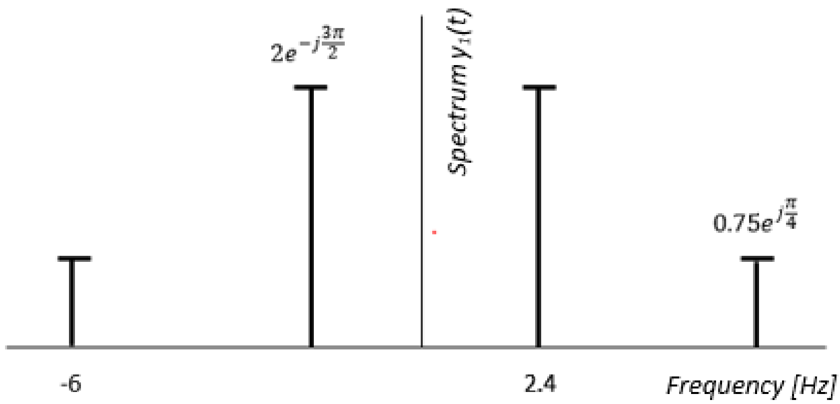

Figure 1: Frequency spectrum of the signal $y_1(t)$ (Exercise 2)

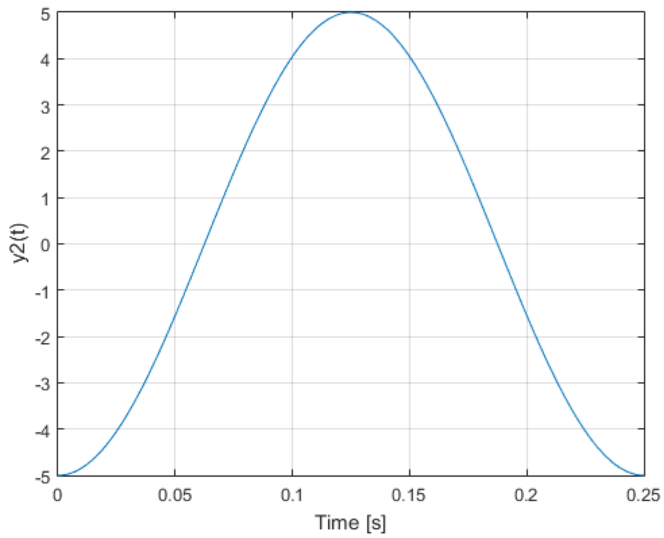

Figure 2: The signal $y_2(t)$ (Exercise 2)

2.5 Frequency spectrum in Matlab

Also in MATLAB, frequency spectra of periodic waveforms can be calculated and plotted. In the next exercise you will calculate the first N Fourier-coefficients of a periodic waveform. Subsequently, you will use these coefficients to generate a 'synthesized' waveform that approximates the original periodic waveform.

Exercise 3 [7 tests]

Consider the periodic signal

$$x(t) = \begin{cases} -1 & \text{for} \quad 0 \le t < \frac{T_0}{4} \\ 2 & \text{for} \quad \frac{T_0}{4} \le t < \frac{T_0}{2} \\ -1 & \text{for} \quad \frac{T_0}{2} \le t < T_0 \end{cases} \tag{12}$$

Calculate the Fourier coefficients α_k by hand for $-10 \le k \le 10$ and for $T_0 = 1$ second and assign the calculated values to the corresponding variables in the template provided in Matlab Grader. Calculate $y(t)$, which is the synthesized approximation of $x(t)$, using $N+1$ Fourier-coefficients and plot $y(t)$. Note that you can perform synthesizing by

$$y(t) = \sum_{k=-N/2}^{N/2} \alpha_k e^{j\frac{2\pi}{T_0}kt}. \tag{13}$$

Make 4 subplots underneath each other for respectively $N = 0, 4, 10, 20$. The time span corresponding to all these plots should be [0, 1] seconds with 101 measuring points. Notice that for higher N, $y(t)$ resembles $x(t)$ more. In case of $N \to \infty$, the resemblance would be perfect.

2.6 Modulation

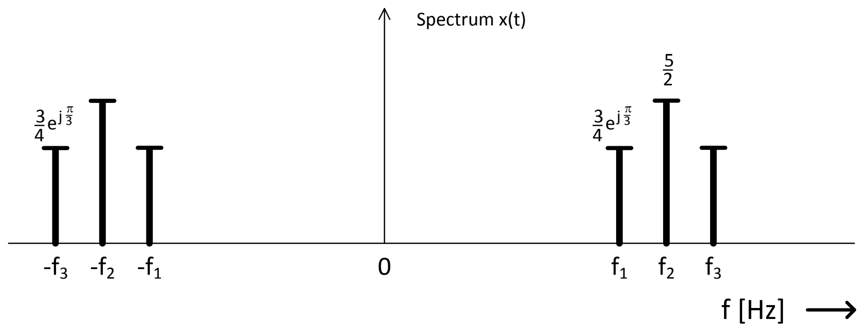

Figure 3: Frequency spectrum of modulated, real signal $x(t)$.

Exercise 4 [5 tests]

Figure 3 represents the frequency spectrum of a modulated, real signal $x(t)$. Express $x(t)$ as the sum of three sinusoids and plot $x(t)$ between 0 and 0.1 seconds. Use Euler to write $x(t)$ as a product which looks like

$$x\left(t\right) = \cos\left(2\pi f_2 t\right) \cdot \left(\mathrm{DC} + A\cos\left(2\pi f_x t + \phi\right)\right), \tag{14}$$

in which $\cos\left(2\pi f_2 t\right)$ is the carrier (or modulation) signal and the term 'DC + $A\cos\left(2\pi f_x t + \phi\right)$' represents the 'message'. Write a script that creates 3 subplots underneath each other. The first subplot should contain the modulated signal $x(t)$ as represented by the frequency spectrum. The second subplot should contain the carrier signal $c(t) = \cos\left(2\pi f_2 t\right)$ and the third subplot should contain the message $m(t) = \mathrm{DC} + A\cos\left(2\pi f_x t + \phi\right)$ .All subplots have the time span [0, 0.1] seconds with 1001 samples. Use the function ylim() to let all three y-axes range from -10 to 10. The frequencies in Figure 3 are $f_1 = 490[Hz]$, $f_2 = 500[Hz]$, and $f_3 = 510[Hz]$.